Kuwait University
Math. & Comp. Sci. Dept.

Math-101
Final Exam

August 02, 2008
Time: 2 hours

Calculators and Mobile Phones are NOT allowed.
Each question counts 4 points.

1) Use the definition of the limit to show that $\lim\limits_{x\to\frac{1}{2}} (4x-5) = -3$.

2) Let $f(x) = \begin{cases} \frac{Ax}{2} + 3 & \text{if } x \le 2 \\ x^2 + \frac{A\sin(x-2)}{x^2-4} & \text{if } x > 2 \end{cases}$. Find the constant A such that $f(x)$ is continuous at $x = 2$.

3) Use the definition of the derivative to find $f'(1)$, where $f(x) = x + \sqrt{x} - 1$.

4) Find the x-coordinate of the point where the tangent line to the curve

$$y = (2x-1)^{1/3} + x^2 + 7$$

is vertical.

5) Let $f(x) = \int_x^{x+3} t(5-t)\,dt$, $x \in \mathbb{R}$. Show that $f(x)$ has a maximum value at $x = 1$.

6) The slope of a curve $y = f(x)$ is given by $m(x) = 2x + \sin x + 1$. Find $f(x)$ knowing that this curve passes through the point $P(0,1)$.

7) Evaluate the following integrals:

a) $\int \left(\frac{\tan\theta + 7}{\cos\theta}\right)^2 d\theta$,

b) $\int_0^2 \frac{x^2}{\sqrt{2x^3+9}}\,dx$.

8) Set up an integral that can be used to find the area of the region bounded by the x-axis and the curve $y = x^3 - 9x$.

9) Set up an integral for the volume of the solid obtained when the region bounded by $y = x^2 + 3$ and $y = 4x$ is revolved about:

a) y-axis,

b) $y = -1$.

10) Find the average value of $f(x) = 3\sqrt{x}$ on $[0,4]$, and determine the number c satisfying the Mean Value Theorem for Integrals.

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1. let $\epsilon > 0$ be given, $|4x-5+3| < \epsilon \Rightarrow 4|x-1/2| < \epsilon$
$\Rightarrow |x-1/2| < \epsilon/4$. let $0 < \delta < \epsilon/4$, $\forall \epsilon > 0, \exists \delta > 0$ s.t.
$0 < |x-1/2| < \delta \Rightarrow |4x-5+3| < \epsilon \Rightarrow \lim_{x \to 1/2} 4x-5 = -3$.

2. $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{Ax}{2} + 3 = A + 3$
$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + \frac{A \sin(x-2)}{(x-2)(x+2)} = 4 + \frac{A}{4}$.
Thus, $A + 3 = 4 + \frac{A}{4} \Rightarrow A = 4/3$.

3. $f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{x + \sqrt{x} - 2}{x-1} = \lim_{x \to 1} \frac{(x-2) + \sqrt{x}}{x-1} \cdot \frac{(x-2) - \sqrt{x}}{(x-2) - \sqrt{x}}$
$= \lim_{x \to 1} \frac{x^2 - 4x + 4 - x}{(x-1)(x-2-\sqrt{x})} = \lim_{x \to 1} \frac{x^2 - 5x + 4}{(x-1)(x-2-\sqrt{x})}$
$= \lim_{x \to 1} \frac{(x-1)(x-4)}{(x-1)(x-2-\sqrt{x})} = \frac{-3}{-2} = 3/2$.

4. $f'(x) = \frac{2}{3}(2x-1)^{-2/3} + 2x$. f is cont. at $x = 1/2$, $\lim_{x \to 1/2} |f'(x)| = \infty$.
The curve has a vertical tangent line $x = 1/2$.

5. $f'(x) = (x+3)[5-(x+3)] - x(5-x) = -6x + 6$.
$f'(x) = 0 \Leftrightarrow x = 1$ and $f''(x) = -6 < 0$. Thus, $f(1)$ is a Max.

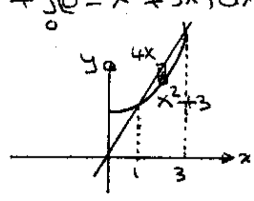
6. $f(x) = \int (2x + \sin x + 1)\, dx = x^2 - \cos x + x + C$. $f(0) = 1 \Rightarrow$
$1 = -1 + C \Rightarrow C = 2$ and $f(x) = x^2 - \cos x + x + 2$.

7. a) $\int \left(\frac{\tan\theta + 7}{\cos\theta}\right)^2 d\theta = \int (\tan\theta + 7)^2 \sec^2\theta \, d\theta \underset{\substack{u = \tan\theta + 7 \\ du = \sec^2\theta \, d\theta}}{=} \frac{1}{3}(\tan\theta + 7)^3 + C$.

b) $\int_0^2 \frac{x^2}{\sqrt{2x^3+9}}\, dx \underset{\substack{u = 2x^3 + 9 \\ du = 6x^2 dx}}{=} \frac{1}{6} \int_9^{25} \frac{du}{\sqrt{u}} = \frac{1}{3} \sqrt{u} \Big|_9^{25} = \frac{2}{3}$.

8. $x^3 - 9x = 0 \Rightarrow x = -3, 0, 3$. $A = \int_{-3}^{0} (x^3 - 9x - 0)\, dx + \int_0^3 (0 - x^3 + 9x)\, dx$

9. $x^2 + 3 = 4x \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$.



a) $V = \int_1^3 2\pi x [4x - (x^2+3)]\, dx$.

b) $V = \int_1^3 \pi [(4x+1)^2 - (x^2+4)^2]\, dx$.

10. $f_{av} = \frac{1}{4} \int_0^4 3\sqrt{x}\, dx = \frac{3}{4} \cdot \frac{2}{3} x^{3/2} \Big|_0^4 = 4$. $3\sqrt{c} = 4 \Rightarrow c = \frac{16}{9}$.